Active Control of Car Suspension Systems using IDA-PBC

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Abstract— **This paper considers the design of active control for car suspension systems using a particular form of energy-based control called Interconnection-and-Damping-Assignment Passivity-Based Control (IDA-PBC). This approach allows one to *shape* the kinetic and potential energy as well as modify the power flow among different components of the system by changing the interconnection and dissipative structure in a meaningful way. Different controller parameterisations are considered to design a class of controllers for active suspension systems.**

I. INTRODUCTION

Car suspension systems are designed to reduce road- and steering-induced vibration, and they also contribute to improve vehicle handling and manoeuvring capabilities [1]. Controlled suspension systems in cars can be classified into active or semi-active. In active systems, a passive spring and damper are interconnected sharing velocity with an actively controlled force actuator that can inject power into the system as well as take power away—dissipation. In the semi-active counterpart, the damper characteristic is controlled; thus energy can only be dissipated at different rates. From a vehicle performance perspective, a controlled suspension system should minimise vertical acceleration at the passenger location and reduce as much as possible tyre deflections in order to secure good steering and manoeuvring handling capabilities—these two objectives are a tradeoff [1].

This paper considers the design of active control for car suspension systems, where the actuator acting in conjunction with the spring and damper us controlled using an energy-based control strategy called Interconnection-and-Damping-Assignment Passivity-Based Control (IDA-PBC) [2]. This control approach allows one to *shape* the kinetic and potential energy as well as modify the power flow among different components of the system by changing the interconnection and dissipative structure of the system. We show that, in our design, the equilibrium point of the unforced closed-loop system is globally asymptotically stable, and with a bounded road disturbance input (vertical displacement and velocity), and that the closed-loop system is input-state-stable (ISS).

In [3], work related to optimal control for this application problem is surveyed. Within such approach, the coefficients of the control cost functional, which are the control tuning parameters, have little physical meaning. With the method proposed in this paper, the physical meaning of the controller coefficients is regained. This paper complements the previous work on semi-active suspension in Port-Hamiltonian form [4], and active suspension in Brayton-Moser form [5].

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II. SET-POINT REGULATION FOR UNDERACTUATED MECHANICAL SYSTEMS IN PHS FORM

In this section, we review the set-point regulation control of underactuated mechanical systems in Port-Hamiltonian form (PHS) using IDA-PBC [2].

A. Open-loop Port-Hamiltonian model

We consider the following Hamiltonian model of a simple mechanical system with added velocity controlled dissipation and external force inputs:

$$\begin{bmatrix}\dot{\mathbf{p}}\\ \dot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & -\mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\frac{\partial \mathcal{H}}{\partial \mathbf{p}}\\ \frac{\partial \mathcal{H}}{\partial \mathbf{q}}\end{bmatrix} - \begin{bmatrix}\frac{\partial \mathcal{D}}{\partial \dot{\mathbf{q}}}\\ \mathbf{0}\end{bmatrix} + \begin{bmatrix}\mathbf{G}\\ \mathbf{0}\end{bmatrix}\mathbf{u}, \tag{1}$$

where $\mathbf{q} \in \mathbb{R}^n$ is the vector of the generalised displacements, $\mathbf{p} \in \mathbb{R}^n$ is the vector of conjugate momenta, $\mathbf{u} \in \mathbb{R}^m$ is the vector of inputs, $\mathbf{G}$ weighs the action of the input on the system and the Hamiltonian is given by

$$\mathcal{H}(\mathbf{p},\mathbf{q}) = \underbrace{\frac{1}{2}\mathbf{p}^{\mathsf{T}}\mathbf{M}^{-1}(\mathbf{q})\mathbf{p}}_{\mathcal{T}(\mathbf{p},\mathbf{q})} + \mathcal{V}(\mathbf{q}). \tag{2}$$

where $\mathbf{M}$ is the generalised mass matrix. The stored kinetic and potential energies in the system are $\mathcal{T}(\mathbf{p},\mathbf{q})$ and $\mathcal{V}(\mathbf{q})$ respectively. The Rayleigh dissipation function $\mathcal{D}(\dot{\mathbf{q}})$ represents the power lost to the environment by the dissipative elements and satisfies the condition $\frac{\partial^{\mathsf{T}}\mathcal{D}}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} \geq 0$ [6]. If the gradient of the dissipation function can be factored as $\frac{\partial \mathcal{D}}{\partial \dot{\mathbf{q}}} = \mathbf{D}\frac{\partial \mathcal{H}}{\partial \mathbf{p}}$, where $\mathbf{D} = \mathbf{D}^{\mathsf{T}} \geq 0$, then we can write (1) in the familiar Port-Hamiltonian form,

$$\begin{bmatrix}\dot{\mathbf{p}}\\ \dot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}-\mathbf{D} & -\mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\frac{\partial \mathcal{H}}{\partial \mathbf{p}}\\ \frac{\partial \mathcal{H}}{\partial \mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{G}\\ \mathbf{0}\end{bmatrix}\mathbf{u}. \tag{3}$$

B. Closed-loop PHS

The objective of the control design is to render the closed-loop system in the following PHS form:

$$\begin{bmatrix}\dot{\mathbf{p}}\\ \dot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{J}-\mathbf{D}_d & -\mathbf{M}_d(\mathbf{q})\mathbf{M}^{-1}(\mathbf{q})\\ \mathbf{M}^{-1}(\mathbf{q})\mathbf{M}_d(\mathbf{q}) & \mathbf{0}\end{bmatrix}\begin{bmatrix}\frac{\partial \mathcal{H}_d}{\partial \mathbf{p}}\\ \frac{\partial \mathcal{H}_d}{\partial \mathbf{q}}\end{bmatrix}, \tag{4}$$

where $\mathbf{J} = -\mathbf{J}^{\mathsf{T}}$ is an additional interconnection term, $\mathbf{D}_d = \mathbf{D}_d^{\mathsf{T}} \geq 0$ is the desired damping and the desired closed-loop Hamiltonian is given by

$$\mathcal{H}_d(\mathbf{p},\mathbf{q}) = \underbrace{\frac{1}{2}\mathbf{p}^{\mathsf{T}}\mathbf{M}_d^{-1}(\mathbf{q})\mathbf{p}}_{\mathcal{T}_d(\mathbf{p},\mathbf{q})} + \mathcal{V}_d(\mathbf{q}), \tag{5}$$

where $\mathbf{M}_d(\mathbf{q}) = \mathbf{M}_d^{\mathsf{T}}(\mathbf{q}) > 0$ is used to shape the kinetic energy $\mathcal{T}(\mathbf{p},\mathbf{q})$. The potential energy $\mathcal{V}_d(\mathbf{q})$ is used to assign

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the desired closed-loop equilibrium, at which $\mathcal{V}_d(\mathbf{q})$ attains its minimum.

C. Matching

To find a control law $\mathbf{u}$, we must *match* the $\dot{\mathbf{p}}$ and $\dot{\mathbf{q}}$ equations in (3) and (4). The $\dot{\mathbf{q}}$ equation is already matched by construction of (4) since

$$\begin{aligned}\dot{\mathbf{q}} &= \frac{\partial \mathcal{H}}{\partial \mathbf{p}} = \mathbf{M}^{-1}(\mathbf{q})\mathbf{p} \\ &= \mathbf{M}^{-1}(\mathbf{q})\mathbf{M}_d(\mathbf{q})\mathbf{M}_d^{-1}(\mathbf{q})\mathbf{p} \\ &= \mathbf{M}^{-1}(\mathbf{q})\mathbf{M}_d(\mathbf{q})\frac{\partial \mathcal{H}_d}{\partial \mathbf{p}}.\end{aligned} \tag{6}$$

The matching of $\dot{\mathbf{p}}$ yields

$$\begin{aligned}\dot{\mathbf{p}} &= -\mathbf{D}\mathbf{M}^{-1}(\mathbf{q})\mathbf{p} - \frac{\partial \mathcal{H}}{\partial \mathbf{q}} + \mathbf{G}\mathbf{u} \\ &= (\mathbf{J} - \mathbf{D}_d)\mathbf{M}_d^{-1}(\mathbf{q})\mathbf{p} - \mathbf{M}_d(\mathbf{q})\mathbf{M}^{-1}(\mathbf{q})\frac{\partial \mathcal{H}_d}{\partial \mathbf{q}}.\end{aligned} \tag{7}$$

Since (7) consists of n equations and $m < n$ unknown control forces, we need to satisfy the following additional $n - m$ constraints to find a solution for $\mathbf{u}$:

$$\mathbf{G}^{\perp}\Big\{\mathbf{D}\mathbf{M}^{-1}(\mathbf{q})\mathbf{p} + (\mathbf{J} - \mathbf{D}_d)\mathbf{M}_d^{-1}(\mathbf{q})\mathbf{p} + \frac{\partial \mathcal{H}}{\partial \mathbf{q}} - \mathbf{M}_d(\mathbf{q})\mathbf{M}^{-1}(\mathbf{q})\frac{\partial \mathcal{H}_d}{\partial \mathbf{q}}\Big\} = \mathbf{0}, \tag{8}$$

where $\mathbf{G}^{\perp}$ is any full-rank left annihilator of $\mathbf{G}$, that is, $\mathbf{G}^{\perp}\mathbf{G} = \mathbf{0}$ and $\operatorname{rank}(\mathbf{G}^{\perp}) = n - m$. If (8) is satisfied, the control law is given by

$$\mathbf{u} = \left(\mathbf{G}^{\mathsf{T}}\mathbf{G}\right)^{-1}\mathbf{G}^{\mathsf{T}}\Big\{\mathbf{D}\mathbf{M}^{-1}(\mathbf{q})\mathbf{p} + (\mathbf{J} - \mathbf{D}_d)\mathbf{M}_d^{-1}(\mathbf{q})\mathbf{p} + \frac{\partial \mathcal{H}}{\partial \mathbf{q}} - \mathbf{M}_d(\mathbf{q})\mathbf{M}^{-1}(\mathbf{q})\frac{\partial \mathcal{H}_d}{\partial \mathbf{q}}\Big\}. \tag{9}$$

We can separate the matching equation, (8), into powers of $\mathbf{p}$, under the assumption that $\mathbf{D}$ and $\mathbf{D}_d$ are independent of $\mathbf{p}$. Furthermore, we assume $\mathbf{J} = \mathbf{J}_0(\mathbf{q}) + \mathbf{J}_1(\mathbf{p}, \mathbf{q})$ where $\mathbf{J}_0(\mathbf{q}) = -\mathbf{J}_0^{\mathsf{T}}(\mathbf{q})$, and $\mathbf{J}_1(\mathbf{p}, \mathbf{q}) = -\mathbf{J}_1^{\mathsf{T}}(\mathbf{p}, \mathbf{q})$ is restricted to depend on $\mathbf{p}$ linearly. Thus, a particular solution of (8) is obtained by solving the following equations:

$$\mathbf{G}^{\perp}\left\{\frac{\partial \mathcal{V}}{\partial \mathbf{q}} - \mathbf{M}_d(\mathbf{q})\mathbf{M}^{-1}(\mathbf{q})\frac{\partial \mathcal{V}_d}{\partial \mathbf{q}}\right\} = \mathbf{0}, \tag{10a}$$

$$\mathbf{G}^{\perp}\left\{\mathbf{D}\mathbf{M}^{-1}(\mathbf{q})\mathbf{p} + (\mathbf{J}_0 - \mathbf{D}_d)\mathbf{M}_d^{-1}(\mathbf{q})\mathbf{p}\right\} = \mathbf{0}, \tag{10b}$$

$$\mathbf{G}^{\perp}\left\{\frac{\partial \mathcal{T}}{\partial \mathbf{q}} + \mathbf{J}_1\mathbf{M}_d^{-1}(\mathbf{q})\mathbf{p} - \mathbf{M}_d(\mathbf{q})\mathbf{M}^{-1}(\mathbf{q})\frac{\partial \mathcal{T}_d}{\partial \mathbf{q}}\right\} = \mathbf{0}. \tag{10c}$$

The control objective reduces to choosing $\mathbf{M}_d(\mathbf{q}) = \mathbf{M}_d^{\mathsf{T}}(\mathbf{q}) > 0$, $\mathcal{V}_d(\mathbf{q}) > 0$, $\mathbf{D}_d = \mathbf{D}_d^{\mathsf{T}} \geq 0$, $\mathbf{J}_0 = -\mathbf{J}_0^{\mathsf{T}}$ and $\mathbf{J}_1(\mathbf{p}) = -\mathbf{J}_1^{\mathsf{T}}(\mathbf{p})$ such that (10) is satisfied.

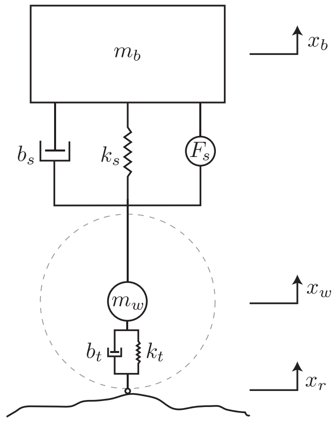

Fig. 1: Idealised quarter-car suspension system.

III. Quarter-car active suspension system

A lumped-parameter model of the quarter-car active suspension system is shown in Figure 1. The quarter chassis mass m_b is supported by spring, k_s, damper, b_s and an active force actuator, F_s. The mass m_w represents the total unsprung mass of the wheel including the tyre, rim, brake rotor and suspension link masses. The road is considered a source of velocity which acts on the wheel via the tyre stiffness, k_t, and the tyre damping coefficient b_t.

The coordinates x_w and x_b are the wheel and body displacements from their respective equilibrium positions, and $x_r(t)$ is the time-varying road position. We define the generalised coordinates $q_1 \triangleq x_w - x_r(t)$ and $q_2 \triangleq x_b - x_w$. Using these coordinates, the kinetic co-energy is given by

$$\mathcal{T}^*(\dot{\mathbf{q}}, t) = \frac{1}{2}m_w(\dot{x}_r(t) + \dot{q}_1)^2 + \frac{1}{2}m_b(\dot{x}_r(t) + \dot{q}_1 + \dot{q}_2)^2, \tag{11}$$

the potential energy is given by

$$\mathcal{V}(\mathbf{q}) = \frac{1}{2}k_t q_1^2 + \frac{1}{2}k_s q_2^2, \tag{12}$$

and the dissipated power is given by

$$\mathcal{D}(\dot{\mathbf{q}}) = \frac{1}{2}b_t \dot{q}_1^2 + \frac{1}{2}b_s \dot{q}_2^2. \tag{13}$$

The conjugate momenta are given by

$$p_1 \triangleq \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = m_w(\dot{x}_r(t) + \dot{q}_1) + m_b(\dot{x}_r(t) + \dot{q}_1 + \dot{q}_2), \tag{14}$$

$$p_2 \triangleq \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = m_b(\dot{x}_r(t) + \dot{q}_1 + \dot{q}_2), \tag{15}$$

where $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathcal{T}^*(\dot{\mathbf{q}}, t) - \mathcal{V}(\mathbf{q})$ is the Lagrangian. The Hamiltonian is defined by $\mathcal{H}(\mathbf{p}, \mathbf{q}) = \mathcal{T}(\mathbf{p}) + \mathcal{V}(\mathbf{q})$ where the kinetic energy is given by $\mathcal{T}(\mathbf{p}) = \frac{1}{2}\mathbf{p}^{\mathsf{T}}\mathbf{M}^{-1}\mathbf{p}$ and

$$\mathbf{M} = \begin{bmatrix} m_w + m_b & m_b \\ m_b & m_b \end{bmatrix}. \tag{16}$$

Note that due to the input $\dot{x}_r(t)$, we have that $\dot{\mathbf{q}} \neq \partial \mathcal{H}/\partial \mathbf{p}$. Indeed, from (14), (15) and the Hamiltonian $\mathcal{H}(\mathbf{p}, \mathbf{q})$, it follows that

$$\dot{q}_1 = \frac{\partial \mathcal{H}}{\partial p_1} - \dot{x}_r(t), \tag{17}$$

$$\dot{q}_2 = \frac{\partial \mathcal{H}}{\partial p_2}. \tag{18}$$

The above differs from (3) due to the modelling hypothesis of considering the road as a velocity input, which still results in a port—an interconnection consisting of power variables.

The momenta dynamics are given by

$$\begin{aligned}\dot{p}_1 &= -\frac{\partial \mathcal{H}}{\partial q_1} - \frac{\partial \mathcal{D}}{\partial \dot{q}_1} \\ &= -\frac{\partial \mathcal{H}}{\partial q_1} - b_t\left(\frac{\partial \mathcal{H}}{\partial p_1} - \dot{x}_r(t)\right),\end{aligned} \tag{19}$$

$$\begin{aligned}\dot{p}_2 &= -\frac{\partial \mathcal{H}}{\partial q_2} - \frac{\partial \mathcal{D}}{\partial \dot{q}_2} + F_s \\ &= -\frac{\partial \mathcal{H}}{\partial q_2} - b_s\frac{\partial \mathcal{H}}{\partial p_2} + F_s.\end{aligned} \tag{20}$$

Equations (17-20) can be compactly written in Port-Hamiltonian form with force and velocity inputs,

$$\begin{bmatrix}\dot{\mathbf{p}}\\ \dot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}-\mathbf{D} & -\mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\frac{\partial \mathcal{H}}{\partial \mathbf{p}}\\ \frac{\partial \mathcal{H}}{\partial \mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{G}\\ \mathbf{0}\end{bmatrix}F_s + \begin{bmatrix}\mathbf{G}_1\\ \mathbf{G}_2\end{bmatrix}\dot{x}_r(t), \tag{21}$$

where $\mathbf{D} = \mathrm{diag}(b_t, b_s)$, $\mathbf{G} = [0,1]^\top$, $\mathbf{G}_1 = [b_t, 0]^\top$ and $\mathbf{G}_2 = [-1, 0]^\top$.

A. *Time-varying state transformation*

To render the open-loop system (21) into a structure similar to that of (3) and to design the control input F_s, we can define a time-varying state transformation

$$\boldsymbol{\zeta} \triangleq \mathbf{p} - \mathbf{G}_1 x_r(t), \tag{22}$$

$$\boldsymbol{\eta} \triangleq \mathbf{q} - \mathbf{G}_2 x_r(t). \tag{23}$$

This results in an open-loop system of the form

$$\begin{bmatrix}\dot{\boldsymbol{\zeta}}\\ \dot{\boldsymbol{\eta}}\end{bmatrix} = \begin{bmatrix}-\mathbf{D} & -\mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\frac{\partial \mathcal{E}}{\partial \boldsymbol{\zeta}}\\ \frac{\partial \mathcal{E}}{\partial \boldsymbol{\eta}}\end{bmatrix} + \begin{bmatrix}\mathbf{G}\\ \mathbf{0}\end{bmatrix}F_s, \tag{24}$$

where

$$\mathcal{E}(\boldsymbol{\zeta}, \boldsymbol{\eta}, t) = \mathcal{W}(\boldsymbol{\zeta}, t) + \mathcal{U}(\boldsymbol{\eta}, t), \tag{25}$$

and

$$\mathcal{W}(\boldsymbol{\zeta}, t) = \frac{1}{2}\begin{bmatrix}\zeta_1 + b_t x_r(t)\\ \zeta_2\end{bmatrix}^\top \mathbf{M}^{-1}\begin{bmatrix}\zeta_1 + b_t x_r(t)\\ \zeta_2\end{bmatrix}, \tag{26}$$

$$\mathcal{U}(\boldsymbol{\eta}, t) = \frac{1}{2}k_t(\eta_1 - x_r(t))^2 + \frac{1}{2}k_s\eta_2^2. \tag{27}$$

Note that the PHS model (24) is in a similar form to (3), except that it has a time-varying storage function, (25).

B. *Control design*

Let the desired closed-loop dynamics be described by following the Port-Hamiltonian System:

$$\begin{bmatrix}\dot{\boldsymbol{\zeta}}\\ \dot{\boldsymbol{\eta}}\end{bmatrix} = \begin{bmatrix}\mathbf{J}_0 - \mathbf{D}_d & -\mathbf{M}_d\mathbf{M}^{-1}\\ \mathbf{M}^{-1}\mathbf{M}_d & \mathbf{0}\end{bmatrix}\begin{bmatrix}\frac{\partial \mathcal{E}_d}{\partial \boldsymbol{\zeta}}\\ \frac{\partial \mathcal{E}_d}{\partial \boldsymbol{\eta}}\end{bmatrix}, \tag{28}$$

where $\mathbf{J}_0 = -\mathbf{J}_0^\top$ is an additional interconnection term, $\mathbf{D}_d = \mathbf{D}_d^\top > 0$ is the desired closed-loop damping, $\mathbf{M}_d$ is the desired closed-loop mass matrix,

$$\mathcal{E}_d(\boldsymbol{\zeta}, \boldsymbol{\eta}, t) = \mathcal{W}_d(\boldsymbol{\zeta}, t) + \mathcal{U}_d(\boldsymbol{\eta}, t), \tag{29}$$

and

$$\mathcal{W}_d(\boldsymbol{\zeta}, t) = \frac{1}{2}\begin{bmatrix}\zeta_1 + b_t x_r(t)\\ \zeta_2\end{bmatrix}^\top \mathbf{M}_d^{-1}\begin{bmatrix}\zeta_1 + b_t x_r(t)\\ \zeta_2\end{bmatrix}, \tag{30}$$

and $\mathcal{U}_d(\boldsymbol{\eta}, t)$ is the desired closed-loop potential energy.

Since (28) has the same form as (4), we can solve (10) to satisfy matching, and then use (9) to find the control law.

This choice of desired closed-loop system is motivated by the fact that the open-loop mass matrix $\mathbf{M}$ is constant. We can trivially satisfy the kinetic energy matching equation (10c) with any constant $\mathbf{M}_d$ and then set $\mathbf{J}_1 = \mathbf{0}$. Then, to satisfy matching, we have only potential energy (10a) and damping (10b) to consider.

Let $\mathbf{M}_d$ have the following parameterisation:

$$\mathbf{M}_d = \begin{bmatrix}a_1 & a_2\\ a_2 & a_3\end{bmatrix}, \tag{31}$$

where a_1, a_2 and a_3 are free parameters that must be chosen such that $\mathbf{M}_d > 0$.

Then, using (10a), $\mathcal{U}_d(\boldsymbol{\eta}, t)$ has a solution given by

$$\mathcal{U}_d(\boldsymbol{\eta}, t) = \frac{k_t m_w}{2(a_1 - a_2)}(\eta_1 - x_r(t))^2 + \Phi(\alpha\eta_1 + \eta_2), \tag{32}$$

where

$$\alpha = \frac{a_1 m_b - a_2(m_w + m_b)}{(a_1 - a_2)m_b}, \tag{33}$$

and $\Phi(\cdot)$ is any free function which we choose to regulate the desired equilibrium point. To ensure that $\mathcal{U}_d(\boldsymbol{\eta}, t)$ is positive definite, we choose $\Phi(z) = \frac{1}{2}k_z(z - z^*)^2$ where $z^* = \alpha x_r^*$ and x_r^* is some nominal road position[1].

Let $\mathbf{D}_d$ and $\mathbf{J}_0$ have the following parameterisation:

$$\mathbf{D}_d = \begin{bmatrix}b_1 & b_2\\ b_2 & b_3\end{bmatrix}, \qquad \mathbf{J}_0 = \begin{bmatrix}0 & -j_0\\ j_0 & 0\end{bmatrix}. \tag{34}$$

Then, using (10b), $\mathbf{D}_d$ has a solution with parameters given by

$$b_1 = (a_1 - a_2)\frac{b_t}{m_w}, \tag{35a}$$

$$b_2 = (a_2 - a_3)\frac{b_t}{m_w} - j_0, \tag{35b}$$

where j_0 and b_3 are free parameters to be chosen such that $\mathbf{D}_d > 0$.

[1]For example, a running average may be used where road preview is not available.

The closed-loop system (28) may be transformed back in the original coordinates, subject to the disturbance input $\dot{x}_r(t)$,

$$\begin{bmatrix}\dot{\mathbf{p}}\\ \dot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{J}_0 - \mathbf{D}_d & -\mathbf{M}_d\mathbf{M}^{-1}\\ \mathbf{M}^{-1}\mathbf{M}_d & \mathbf{0}\end{bmatrix}\begin{bmatrix}\frac{\partial \mathcal{H}_d}{\partial \mathbf{p}}\\ \frac{\partial \mathcal{H}_d}{\partial \mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{G}_1\\ \mathbf{G}_2\end{bmatrix}\dot{x}_r(t), \tag{36}$$

where

$$\mathcal{H}_d(\mathbf{p},\mathbf{q},t) = \frac{1}{2}\mathbf{p}^\top\mathbf{M}_d^{-1}\mathbf{p} + \mathcal{V}_d(\mathbf{q},t), \tag{37}$$

and

$$\mathcal{V}_d(\mathbf{q},t) = \frac{k_t m_w}{2(a_1-a_2)}q_1^2 + \Phi\left(\alpha(x_r(t)+q_1)+q_2\right). \tag{38}$$

IV. Stability properties of the closed loop

In this section, we follows the approach in [7] to study the stability properties of the PHS in the framework of ISS theory. We consider three special cases of the controller proposed in the previous section:

- *Controller 1.* Active damping injection
  - $\mathbf{M}_d = \mathbf{M} \implies \mathcal{T}_d(\mathbf{p}) = \mathcal{T}(\mathbf{p})$
  - $\alpha = 0, k_z = k_s \implies \mathcal{V}_d(\mathbf{q},t) = \mathcal{V}(\mathbf{q})$
  - Choose b_2 (via j_0) and b_3 as free parameters
- *Controller 2.* Energy shaping and damping injection preserving location of open-loop equilibrium
  - $\alpha = 0 \iff a_2 = \frac{m_b}{m_w+m_b}a_1$
  - $b_2 = 0 \iff j_0 = \frac{b_t}{m_w}(a_2 - a_3)$
  - Choose a_1, a_3, b_3, k_z as free parameters
- *Controller 3.* Energy shaping and damping injection with inertial decoupling
  - $\alpha = 1 \iff a_2 = 0$
  - $b_2 = 0 \iff j_0 = \frac{b_t}{m_w}(a_2 - a_3)$
  - Choose a_1, a_3, b_3, k_z as free parameters

The next proposition considers the stability of the closed-loop system using either controller 1 or controller 2 .

Proposition 1: Consider the system (36) with a desired potential energy function $\mathcal{V}_d(\mathbf{q})$ that satisfies

$$\left|\frac{\partial \mathcal{V}_d(\mathbf{q})}{\partial \mathbf{q}}\right| \le \chi(q) \tag{39}$$

where $\chi \in \mathcal{K}_\infty$. Then,

i) The equilibrium point of (36) with $\dot{x}_r(t) = 0$ is globally asymptotically stable.

ii) The control system (36) with input $\dot{x}_r(t)$ is input-state-stable provided that $\left|\frac{\partial^2\mathcal{V}_d(\mathbf{q})}{\partial \mathbf{q}^2}\right| \le k_d$, with k_d a positive constant.

Proof: i) We use the Hamiltonian $\mathcal{H}_d$ as a Lyapunov candidate function. The orbital derivative of $\mathcal{H}_d$ along the solution of (36) yields

$$\dot{\mathcal{H}}_d = -\mathbf{p}^\top\mathbf{M}_d^{-1}\mathbf{D}_d\mathbf{M}_d^{-1}\mathbf{p} \le 0,$$

which ensures Lyapunov stability. Asymptotic stability follows from the Invariance Principle and the fact that the maximum invariant set in $\mathcal{S} = \{(\mathbf{p},\mathbf{q})|\mathcal{H}_d(\mathbf{p},\mathbf{q}) = 0\} = (\mathbf{0},\bar{\mathbf{q}})$. Since $\mathcal{H}_d$ is proper, the stability property is global.

ii) To prove ISS we use a ISS-Lyapunov candidate function as follows

$$\mathcal{Q}(\mathbf{p},\mathbf{q}) = \frac{1}{2}\mathbf{p}^T\mathbf{M}_d^{-1}\mathbf{p} + \mathcal{V}_d(\mathbf{p}) + \epsilon\mathbf{p}^\top\mathbf{M}_d^{-1}\frac{\partial \mathcal{V}_d}{\partial \mathbf{q}}, \tag{40}$$

which is positive definite for a sufficiently small constant $\epsilon > 0$. The derivative of $\mathcal{Q}$ with respect to time yields

$$\begin{aligned}\dot{\mathcal{Q}} &= \frac{\partial^\top\mathcal{Q}}{\partial\mathbf{p}}\dot{\mathbf{p}} + \frac{\partial^\top\mathcal{Q}}{\partial\mathbf{q}}\dot{\mathbf{q}} = [\mathbf{M}_d^{-1}\mathbf{p} + \epsilon\mathbf{M}_d^{-1}\frac{\partial\mathcal{V}_d}{\partial\mathbf{q}}]^\top[-\mathbf{M}_d\\ &\quad \mathbf{M}^{-1}\frac{\partial\mathcal{V}_d}{\partial\mathbf{q}} + (\mathbf{J}_0 - \mathbf{D}_d)\mathbf{M}_d^{-1}\mathbf{p}] + [\frac{\partial\mathcal{V}_d}{\partial\mathbf{q}} + \epsilon\frac{\partial^2\mathcal{V}_d}{\partial\mathbf{q}^2}\\ &\quad \mathbf{M}_d^{-1}\mathbf{p}]^\top\mathbf{M}^{-1}\mathbf{p} + [\mathbf{M}_d^{-1}\mathbf{p} + \epsilon\mathbf{M}_d^{-1}\frac{\partial\mathcal{V}_d}{\partial\mathbf{q}}]^\top\mathbf{G}_1 u +\\ &\quad [\frac{\partial\mathcal{V}_d}{\partial\mathbf{q}} + \epsilon\frac{\partial^2\mathcal{V}_d}{\partial\mathbf{q}^2}\mathbf{M}_d^{-1}\mathbf{p}]^\top\mathbf{G}_2 u\\ &= -\mathbf{p}^\top\mathbf{F}\mathbf{p} - \frac{\partial^\top\mathcal{V}_d}{\partial\mathbf{q}}\epsilon\mathbf{M}^{-1}\frac{\partial\mathcal{V}_d}{\partial\mathbf{q}} + \frac{\partial^\top\mathcal{V}_d}{\partial\mathbf{q}}\epsilon\mathbf{M}^{-1}\\ &\quad (\mathbf{J}_0 - \mathbf{D}_d)\mathbf{M}_d^{-1}\mathbf{p} + \mathbf{p}^\top\mathbf{M}_d^{-1}\mathbf{G}_1 u + \frac{\partial^\top\mathcal{V}_d}{\partial\mathbf{q}}\epsilon\mathbf{M}_d^{-1}\\ &\quad \mathbf{G}_1 u\frac{\partial^\top\mathcal{V}_d}{\partial\mathbf{q}}\mathbf{G}_2 u + \mathbf{p}^\top\epsilon\mathbf{M}_d^{-1}\frac{\partial^2\mathcal{V}_d}{\partial\mathbf{q}^2}\mathbf{G}_2 u,\end{aligned} \tag{41}$$

where we use the matrix $\mathbf{F} = \mathbf{M}_d^{-1}\mathbf{D}_d\mathbf{M}_d^{-1} - \epsilon\mathbf{M}_d^{-1}\frac{\partial^2\mathcal{V}_d}{\partial\mathbf{q}^2}\mathbf{M}^{-1}$. Given that the Hessian of the potential energy is bounded, then $\mathbf{F}$ is positive definite for a sufficiently small ϵ. Consequently, its symmetric part $\mathbf{F}_{sy} = \frac{\mathbf{F}+\mathbf{F}^\top}{2}$ is also positive definite. Using these matrices and $\mathbf{E} = \frac{\epsilon}{2}\mathbf{M}^{-1}(\mathbf{J}_0 - \mathbf{D}_d)\mathbf{M}_d^{-1}$, we can write

$$\begin{aligned}\dot{\mathcal{Q}} &\le -[\mathbf{p}^\top\ \frac{\partial^\top\mathcal{V}_d}{\partial\mathbf{q}}]\begin{bmatrix}\mathbf{F}_{sy} & \mathbf{E}^\top\\ \mathbf{E} & \epsilon\mathbf{M}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{p}\\ \frac{\partial\mathcal{V}_d}{\partial\mathbf{q}}\end{bmatrix} +\\ &\quad \left(|\mathbf{M}_d^{-1}||\mathbf{G}_1| + \epsilon|\mathbf{M}_d^{-1}|\left|\frac{\partial^2\mathcal{V}_d}{\partial\mathbf{q}^2}\right||\mathbf{G}_2|\right)|\mathbf{p}||u| +\\ &\quad \left(\epsilon|\mathbf{M}_d^{-1}||\mathbf{G}_1| + |\mathbf{G}_2|\right)\left|\frac{\partial\mathcal{V}_d}{\partial\mathbf{q}}\right||u|\\ &\le -c_1\left(|\mathbf{p}|^2 + \left|\frac{\partial\mathcal{V}_d}{\partial\mathbf{q}}\right|^2\right) + c_2|\mathbf{p}||u| + c_3\left|\frac{\partial\mathcal{V}_d}{\partial\mathbf{q}}\right||u|\end{aligned} \tag{42}$$

for some positive constants c_1, c_2 and c_3. Using the fact that for vectors $\mathbf{v}$ and $\mathbf{w}$ and $a, b \in \mathbb{R}_{>0}$

$$-a|\mathbf{v}|^2 + b|\mathbf{v}||\mathbf{w}| \le -\frac{a}{2}|\mathbf{v}|^2 + \frac{b^2}{2a}|\mathbf{w}|^2,$$

it follows that

$$\dot{\mathcal{Q}} \le -\frac{c_1}{2}\left(|\mathbf{p}|^2 + \left|\frac{\partial\mathcal{V}_d}{\partial\mathbf{q}}\right|^2\right) + \frac{c_2^2 + c_3^2}{2c_1}|u|^2, \tag{43}$$

which proves that $\mathcal{Q}$ is an ISS-Lyapunov function since $|\frac{\partial\mathcal{V}_d}{\partial\mathbf{q}}| \le \chi(q)$. ■

Remark 1. The desired Hamiltonian in closed loop does not qualify as an ISS-Lyapunov function. However, the addition of a cross term, as done in the previous proposition, produces a suitable candidate ISS-Lyapunov function—see for example [8] in the framework of ISS theory or [9] in the context of strictly Lyapunov function for PH systems.

The next proposition considers the stability of the closed-loop system using controller 3.

Proposition 2: Consider the system (36) with a desired potential energy function

$$\mathcal{V}_d(\mathbf{q},t) = \frac{k_1}{2}q_1^2 + \frac{k_2}{2}\left(x_r(t) + q_1 + q_2\right)^2, \qquad (44)$$

which follows from (38) by choosing $\Phi(r) = \frac{1}{2}k_2 r^2$ with $k_2 \in \mathbb{R}_{>0}$, $\alpha = 1$, and $k_1 = \frac{k_t m_w}{2(a_1 - a_2)}$. Then,

i) The equilibrium point of (36) with $\dot{x}_r(t) = 0$ is globally asymptotically stable.
ii) The control system (36) with inputs $\dot{x}_r(t)$ and $x_r(t)$ is input-state-stable.

Proof: i) The system (36) can be written as

$$\begin{bmatrix}\dot{\mathbf{p}}\\ \dot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{J}_0 - \mathbf{D}_d & -\mathbf{M}_d\mathbf{M}^{-1}\\ \mathbf{M}^{-1}\mathbf{M}_d & \mathbf{0}\end{bmatrix}\begin{bmatrix}\frac{\partial\mathcal{P}}{\partial\mathbf{p}}\\ \frac{\partial\mathcal{P}}{\partial\mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{G}_3 & \mathbf{G}_1\\ 0 & \mathbf{G}_2\end{bmatrix}\begin{bmatrix}x_r(t)\\ \dot{x}_r(t)\end{bmatrix}, \qquad (45)$$

where

$$\mathcal{P}(\mathbf{p},\mathbf{q}) = \frac{1}{2}\mathbf{p}^{\mathsf{T}}\mathbf{M}_d^{-1}\mathbf{p} + \frac{1}{2}\mathbf{q}^{\mathsf{T}}\mathbf{K}\mathbf{q}, \qquad (46)$$

$$\mathbf{K} = \begin{bmatrix}k_1 + k_2 & k_2\\ k_2 & k_2\end{bmatrix} = \mathbf{K}^{\mathsf{T}} > 0,$$

and $\mathbf{G}_3 = -k_2\mathbf{M}_d\mathbf{M}^{-1}$. Using $\mathcal{P}$ as a Lyapunov candidate function and computing its time derivative along the solution of (45) it yields

$$\dot{\mathcal{P}} = -\mathbf{p}^{\mathsf{T}}\mathbf{M}_d^{-1}\mathbf{D}_d\mathbf{M}_d^{-1}\mathbf{p} \leq 0,$$

which proves Lyapunov stability. Asymptotic stability follows from the Invariance Principle and the fact that the maximum invariant set in $\mathcal{S} = \{(\mathbf{p},\mathbf{q})|\mathcal{P}(\mathbf{p},\mathbf{q}) = 0\} = (\mathbf{0},\mathbf{0})$. Since $\mathcal{P}$ is proper, the stability property is global.

ii) ISS of the system follows from GAS and noting that the system is linear [10], [11]. Indeed, the system can be written as

$$\begin{bmatrix}\dot{\mathbf{p}}\\ \dot{\mathbf{q}}\end{bmatrix} = \mathbf{A}\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix} + \mathbf{B}\mathbf{u}(t) \qquad (47)$$

with

$$\mathbf{A} = \begin{bmatrix}(\mathbf{J}_0 - \mathbf{D}_d)\mathbf{M}_d^{-1} & -\mathbf{M}_d\mathbf{M}^{-1}\mathbf{K}\\ \mathbf{M}^{-1} & \mathbf{0}\end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix}\mathbf{G}_3 & \mathbf{G}_1\\ 0 & \mathbf{G}_2\end{bmatrix}$$

Since the system is GAS, then $\mathbf{A}$ is Hurwitz and the trajectory solution $\varphi(t)$ can be written as

$$\varphi(t) = e^{\mathbf{A}t}\begin{bmatrix}\mathbf{p}(0)\\ \mathbf{q}(0)\end{bmatrix} + \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Using the bound $|e^{\mathbf{A}t}| \leq ce^{-\lambda t}$, for appropriate positive constants c and λ, we can write

$$\begin{aligned}|\varphi(t)| &\leq ce^{-\lambda t}\left|\begin{bmatrix}\mathbf{p}(0)\\ \mathbf{q}(0)\end{bmatrix}\right| + \int_0^t ce^{-\lambda t}|\mathbf{B}||\mathbf{u}(\tau)|d\tau\\ &\leq ce^{-\lambda t}\left|\begin{bmatrix}\mathbf{p}(0)\\ \mathbf{q}(0)\end{bmatrix}\right| + \frac{c|\mathbf{B}|}{\lambda}\sup_{0\leq\tau\leq t}|\mathbf{u}(\tau)|,\end{aligned}$$

which proves bounded-input-bounded-state stability and ISS. ■

V. Case Study

We consider the design of an active suspension controller for a *scale* model quarter-car rig with parameters given in the Appendix. The simulated scenario consists of a 1cm road amplitude disturbance at 0.5 seconds.

Figure 2 shows the open-loop response of the system. Figure 3 shows the response of the active damping controller

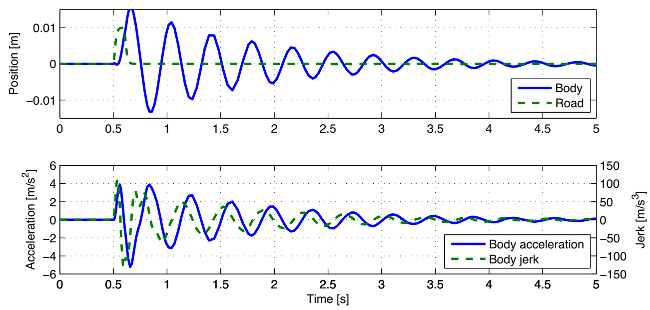

Fig. 2: Open-loop response.

(controller 1). This scheme is active since the j_0 term, which implements $b_2 \neq 0$, requires the controller to exchange power with the system, and therefore it is not a semi-active suspension controller.

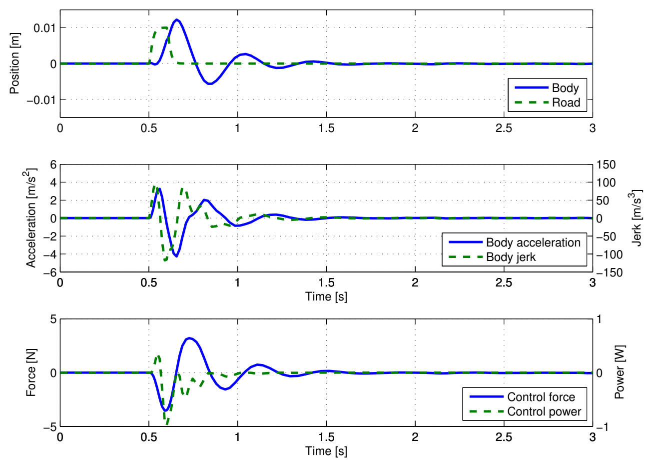

Fig. 3: Active damping injection; $j_0 = -11.5$, $b_3 = 30$.

Figure 4 shows the energy shaping and damping injection controller which preserves the location of the open-loop equilibrium (controller 2). The chassis and wheel systems

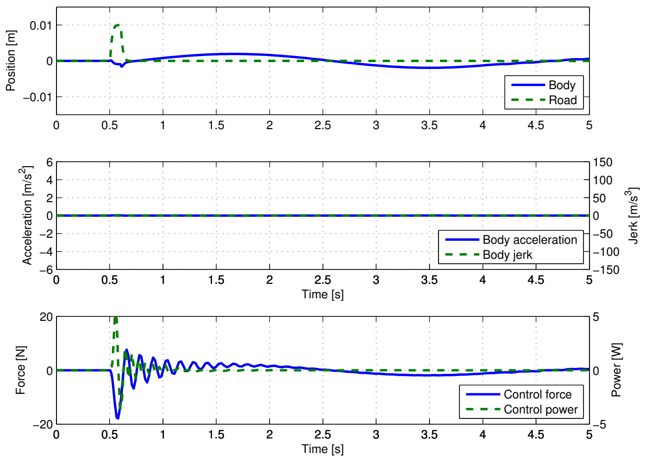

Fig. 4: Energy shaping and damping injection preserving location of open-loop equilibrium; $a_1 = 24.97$, $a_2 = 17.3$, $a_3 = 17.3$, $b_2 = 0$, $b_3 = 0.109$, $k_z = 1$.

are almost decoupled due to the low closed-loop damping b_3 and stiffness k_z. This response has a long settling time.

Figure 5 shows the energy shaping and damping injection controller which decouples the inertia (controller 3). This scheme actively controls the chassis position with respect to an inertial frame.

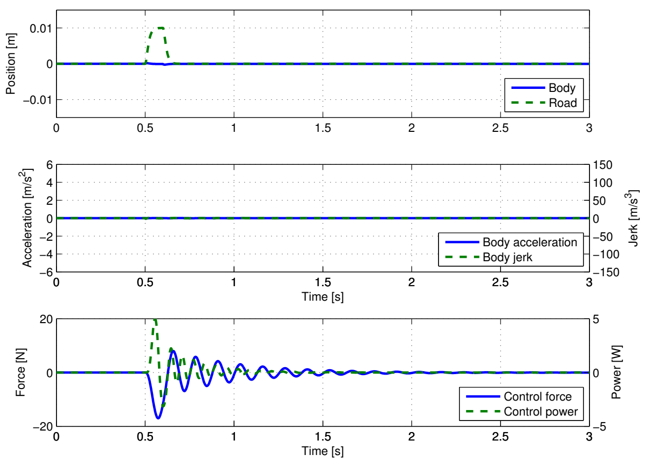

Fig. 5: Energy shaping and damping injection with inertial decoupling; $a_1 = 3.45$, $a_3 = 0.105$, $a_2 = 0$, $b_2 = 0$, $b_3 = 196.5$, $k_z = 1107.4$.

VI. CONCLUSIONS

This paper considers the design of active control for car suspension systems using a particular form of energy-based control. The results complement previous work in the literature. We analyse three controllers, namely, active damping injection and energy shaping with both coupled and decoupled inertias. We show input to state stability for the three controllers and present simulation results based on a scale model of a quarter-car suspension system.

The proposed controller is tuned in terms of parameters that have a clear physical meaning, in terms of mass, damping and spring stiffness coefficients. The comparison between the different controllers suggests that controller 3, namely, energy shaping with inertial decoupling, provides the best performance in terms of minimising vertical acceleration and jerk.

APPENDIX

The suspension parameters are found in Table I.

TABLE I: Plant parameters.

Parameter	Value
m_w	1 kg
m_b	2.45 kg
k_t	2500 N/m
k_s	980 N/m
b_t	5 Ns/m
b_s	7.5 Ns/m

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